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Electromagnetic moments of quasi-stable particle

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We deal with the problem of assigning electromagnetic moments to a quasi-stable particle (i.e., a particle with mass located at particle's decay threshold). In this case, an application of a small external electromagnetic field changes the energy in a non-analytic way, which makes it difficult to assign definitive moments. On the example of a spin-1/2 field with mass M_* interacting with two fields of masses M and m, we show how a conventionally defined magnetic dipole moment diverges at $M_* = M + m$. We then show that the conventional definition makes sense only when the values of the applied magnetic field B satisfy $|eB|/2M_* \ll |M_* - M - m|$. We discuss implications of these results to existing studies in electroweak theory, chiral effective-field theory, and lattice QCD.

Electromagnetic (e.m.) moments of a particle are determined through observations of the particle's behavior in an applied electromagnetic field. For example, the magnetic moment is measured by observing the spin precession in a magnetic field. In doing so, one assumes that the uniform magnetic field $\vec{B}$ induces a linear response in the energy:

$$\Delta E = -\vec{\mu} \cdot \vec{B}, \tag{1}$$

with $\vec{\mu}$ being the magnetic moment. This method works perfectly well for stable particles (electron, proton), as well as for many unstable particles (muon, neutron, etc.), which live long enough for their spin precession to be observed. In this letter we examine the case of a "quasi-stable" particle, i.e., a particle with mass M_* that could decay into two (for simplicity) particles with masses M and m, such that

$$M_* = M + m\,. \tag{2}$$

It turns out that applying the magnetic field in this situation does not lead to a polynomial energy shift but to a response which is non-analytic in B, typically $\Delta E \sim |\vec{B}|^{1/2}$. The square-root behavior is characteristic for the particle-production cut. In a more general situation, when $M_* \approx M + m$, a polynomial expansion in B can be made as long as

$$|\vec{B}| \times [\text{magneton}] \ll |M_* - M - m|, \tag{3}$$

which thus becomes a condition for the magnetic moment to be observable.

We do not yet know of examples in nature where the masses of particles would be tuned to such an extent that the condition Eq. (3) would be violated. For example, the neutron mass is less than 1 MeV above the threshold ($M_n - M_p - m_e \approx 0.8$ MeV), but this number is huge when compared to any reasonable value of the magnetic field measured in units of nuclear magneton: $\mu_N \simeq 3 \times 10^{-14}$ MeV/Tesla. Nevertheless, situations where the condition Eq. (3) is violated are sometimes encountered in theoretical studies. In the studies of the W-boson's magnetic and quadrupole moments as a function of bottom- and top-quark masses m_b and m_t, a singularity at $m_b + m_t = M_W$ arises from the $b\bar{t}$ (or $t\bar{b}$) loop contributions. This singularity was reported firtstly in [1, 2] at a time when the value of m_t was not known yet. In lattice Quantum Chromodynamics (QCD), the e.m. moments of hadron are computed for various values of light quark masses and, as calculations based on chiral perturbation theory show, cups and singularities arise too [3, 4]. In this work we find that the singularities arise in the region where the electromagnetic moments are ill-defined, because the condition Eq. (3) is not satisfied.

Our findings are best demonstrated on a simple toy model of three fields: a scalar φ and two Dirac spinors ψ and Ψ, interacting via the Yukawa type of coupling:

$$\mathcal{L}_{\text{int}} = g\left(\overline{\Psi}\,\psi\,\varphi + \overline{\psi}\,\Psi\,\varphi^*\right), \tag{4}$$

with $g \ll 1$, a small coupling constant. We denote the masses of φ, ψ and Ψ respectively as: m, M, and M_*, and will later on focus on the region specified by Eq. (2).

Suppose the field Ψ, as well as one of the other two fields, has an electric charge e, and couples minimally to electromagnetism. We look for its anomalous magnetic moment (a.m.m.) κ_* at leading order in the coupling g. Depending on whether φ or ψ is charged we ought to consider the electromagnetic vertex corrections shown in Fig. 1, and obtain (unprimed: φ charged, ψ neutral, or primed: ψ charged, φ neutral):

$$\kappa_* = \frac{2g^2}{(4\pi)^2}\int_0^1 dx\,\frac{-(r+x)\,x(1-x)}{x\mu^2 - x(1-x) + (1-x)r^2}, \tag{5}$$

$$\kappa'_* = \frac{2g^2}{(4\pi)^2}\int_0^1 dx\,\frac{(r+x)(1-x)^2}{x\mu^2 - x(1-x) + (1-x)r^2}, \tag{6}$$

where $r = M/M_*$, $\mu = m/M_*$.

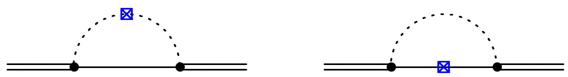

Figure 1: One-loop electromagnetic vertex corrections. Double lines, single and dotted lines denote the propagators of Ψ, ψ, and φ, respectively. Dots denote the Yukawa coupling and rectangles the minimal electromagnetic coupling.

We have checked that for $M_* = -M = M_N$ and $m = m_\pi$ being respectively the mass of the nucleon and the pion, these expressions reproduce results of the meson theory (the same result also arises in chiral perturbation theory at next-to-leading order [5]). The minus sign in front of M appears due to the pseudo-scalar nature of pion.

At $M_* = m + M$ (or, $1 = \mu + r$), the denominator in the integrands takes the form $[x\mu - (1-x)r]^2$, which leads to an essential singularity in these expressions for any positive μ and r. This can explicitly be seen, for instance, in Fig. 2 where κ_* is plotted as a function of μ. If φ is pseudo-scalar, M flips the sign in these expressions, and the singularity is replaced by a cusp.

The singularity is clearly unphysical, since an infinite value of the magnetic moment would correspond to a infinite-energy response to an external magnetic field. To find the correct answer we consider the self-energy of the Ψ-field in a constant electromagnetic field, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \text{const}$. Calculations of this sort have been done before, most notably by Sommerfield and Schwinger [6, 7] as a technique to obtain the correction term of order $\alpha_{\text{em}}^2 \simeq (1/137)^2$ to the electron's a.m.m..

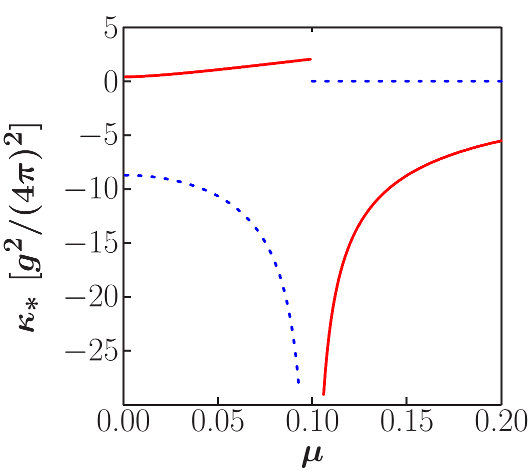

Figure 2: The anomalous magnetic moment κ_* of Ψ-field as function of φ-field mass μ, at fixed value $r = 0.9$. The red (solid) curve shows the real part and the blue (dashed) curve the imaginary part of κ. (The sign of the imaginary part is determined by the $i\varepsilon$ prescription.)

To cast this technique into a modern field-theoretic language, we introduce the sources Θ, and j for the fields Ψ, ψ and φ, respectively, and write down the generating functional of the theory,

$$\begin{aligned} Z[\Theta, , j; A] \ &= \ \exp\Big\{ -g\int d^4z \,\Big(\frac{\delta^3}{\delta j^*(z)\, \delta(z)\, \delta\Theta(z)} \\ &+ \ \frac{\delta^3}{\delta j(z)\, \delta\bar{\Theta}(z)\, \delta(z)} \Big) \Big\} \exp\Big[-\int d^4x\, d^4y \\ &\times \ \bar{\Theta}(x)\, S(x-y;A)\, \Theta(y) + \ldots \Big], \end{aligned} \tag{7}$$

where

$$S(x-y;A) = \left[i\gamma^\mu \tfrac{\partial}{\partial x^\mu} - eA_\mu(x)\gamma^\mu - M_* \right]^{-1} \delta^{(4)}(x-y)\,. \tag{8}$$

is the propagator of a charged Dirac particle in the presence of an e.m. field. We then calculate the energy shift induced by the Ψ-field self-energy correction in the presence of a constant e.m. field. The dependence on the e.m. field comes in the form of the $\gamma^\mu\gamma^\nu F_{\mu\nu}$ structure sandwiched between the free Ψ-field states. When the electric contribution is zero, this structure simply yields the projection of the magnetic field onto the spin direction.

The resulting energy-shift, to leading order in g, is for the two cases given by:

$$\begin{aligned} \Delta\tilde{E} \ &= \ \frac{g^2}{(4\pi)^2} \int_0^1 dx\, (r+x) \qquad (9) \\ &\times \ \ln\Big[1 + \frac{x(1-x)\,\tilde{B}}{x\mu^2 - x(1-x) + (1-x)r^2} \Big], \\ \Delta\tilde{E}' \ &= \ \frac{g^2}{(4\pi)^2} \int_0^1 dx\, (r+x) \qquad (10) \\ &\times \ \ln\Big[1 - \frac{(1-x)^2\,\tilde{B}}{x\mu^2 - x(1-x) + (1-x)r^2} \Big], \end{aligned}$$

where the following dimensionless variables are used:

$$\tilde{B} = \frac{eB_z}{M_*^2}\,, \quad \Delta\tilde{E} = \frac{\Delta E}{M_*} + \frac{1}{2}\tilde{B}\,, \tag{11}$$

with B_z the projection of the magnetic field on the spin direction. The quantity $\Delta\tilde{E}$ is the energy shift (in units of M_*) due to the a.m.m. effect. In the following we will discuss the unprimed contribution, the primed one can be obtained analogously.

The e.m. field is assumed to be small in comparison with the mass-scale of particles, and therefore some terms which are higher-order in $\tilde{B}^2$ can be neglected. Nevertheless, one can still see that a naive perturbative expansion in $\tilde{B}$ does not always work. In the naive expansion, one finds

$$\Delta\tilde{E} = -\frac{\kappa_*}{2}\tilde{B} + \ldots, \tag{12}$$

with κ_* given by Eq. (5), which recovers the conventional result. However, around the (in)stability threshold

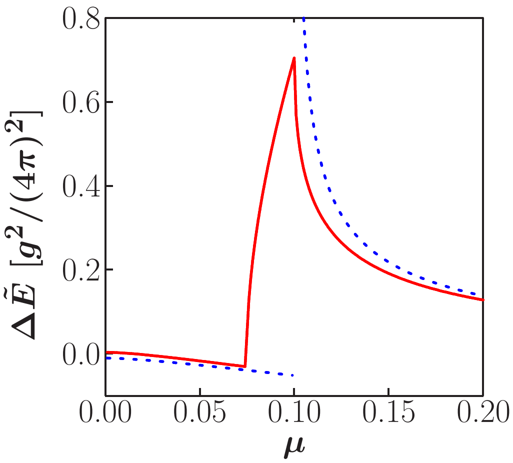

Figure 3: The real part of the energy shift $\Delta\tilde{E}$ as function of μ for a fixed magnetic field strength $\tilde{B}$. The red (solid) curve is obtained from Eq. (9) while the blue (dotted) from Eq. (12) and Eq. (5). The parameters are chosen $r = 0.9$ and $\tilde{B} = 0.05$.

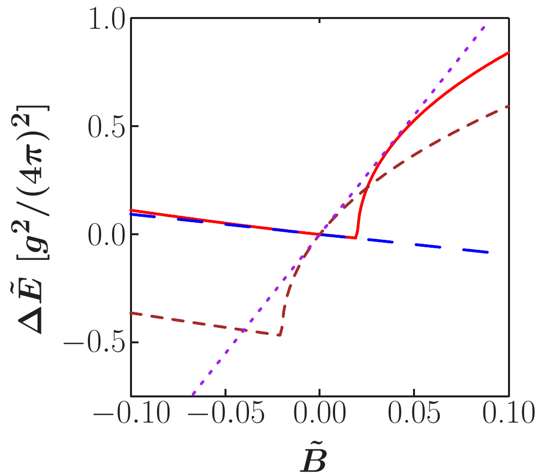

Figure 4: The real part of the energy shift $\Delta\tilde{E}$ as function of the magnetic field $\tilde{B}$ for a fixed μ. The parameters are chosen as $r = 0.9$. The red (solid) curve is obtained from Eq. (9) and the blue (long-dashed) from Eq. (12) with Eq. (5) for $\mu = 0.09$. The brown (medium-dashed) curve is obtained from Eq. (9) while the purple (short-dashed) curve from Eq. (12) for $\mu = 0.11$.

$M_* = M + m$, the naive expansion breaks down, as can be seen from Fig. 3 where we plot the energy shift Eq. (9) compared to the result of the naive perturbative expansion: Eq. (12) with Eq. (5). It is clear that the two results are very different around the threshold which here is at $\mu = 0.1$. The size of the region where the two results are different is proportional to the strength of the magnetic field.

In Fig. 4 we again compare the perturbative and non-perturbative results, but now as a function of the magnetic-field strength. The masses are fixed such that the Ψ particle is stable for solid and long-dashed curves and unstable for medium- and short-dashed curves. In either situation there is a kink appearing at some value of the magnetic field, which indicates the crossing over the decay threshold. When Ψ is quasi-stable, $\mu + r = 1$, the kink appears at $B = 0$, which makes it impossible to define the moments as derivatives of the energy response with respect to the e.m. field.

Integration over the Feynman-parameter x in Eq. (9) yields more insight into the non-analytic dependence on the e.m. field. The result can be written as

$$\Delta\tilde{E} = \frac{g^2}{(4\pi)^2}\left\{(r+\alpha)\,(\mathit{\Omega}+\mathcal{A}) - \left[(r+\alpha)\,(\mathit{\Omega}+\mathcal{A})\right]_{\tilde{B}=0}\right\}, \tag{13}$$

where $\mathit{\Omega}$ is non-analytic in $\tilde{B}$:

$$\mathit{\Omega} \;=\; \lambda\,\ln\frac{(\alpha+\lambda)(\beta+\lambda)}{(\alpha-\lambda)(\beta-\lambda)}, \tag{14}$$

with

$$\begin{aligned} \alpha &= \frac{1}{2(1-\tilde{B})}\left(1+r^2-\mu^2-\tilde{B}\right), \\ \beta &= \frac{1}{2(1-\tilde{B})}\left(1-r^2+\mu^2-\tilde{B}\right), \\ \lambda &= \left[\alpha^2 - r^2/(1-\tilde{B})\right]^{1/2}. \end{aligned} \tag{15}$$

while the analytic terms are contained in

$$\begin{aligned} \mathcal{A} &= -2+\beta\ln\mu^2+\alpha\ln r^2 \\ &\quad - \frac{\mu^2(1-\ln\mu^2)-r^2(1-\ln r^2)}{2(\alpha+r)(1-\tilde{B})}. \end{aligned} \tag{16}$$

From the expression for $\mathit{\Omega}$ we can readily see that a Taylor expansion in B only make sense when the condition of Eq. (3) is satisfied.

The masses of particles are rarely tuned to the extent that the condition Eq. (3) is in danger. One field of applications where one does need to pay attention is lattice QCD. In modern lattice studies the e.m. moments of hadrons can directly be accessed using the background e.m. field method [8]. However, the field strength cannot be arbitrarily small, the periodicity condition poses a lower bound. In the case of magnetic field the bound is: $eB \geq 2\pi/(a^2 L)$, or in best case [9]: $eB \geq 2\pi/(aL)^2$, with length a and integer L being respectively the lattice spacing and size. For typical modern lattices the lowest possible value of the magnetic field can be as large as 10^{14} Tesla. Certainly in such strong e.m. fields the problem raised here becomes relevant and should be studied on a case-by-case basis.

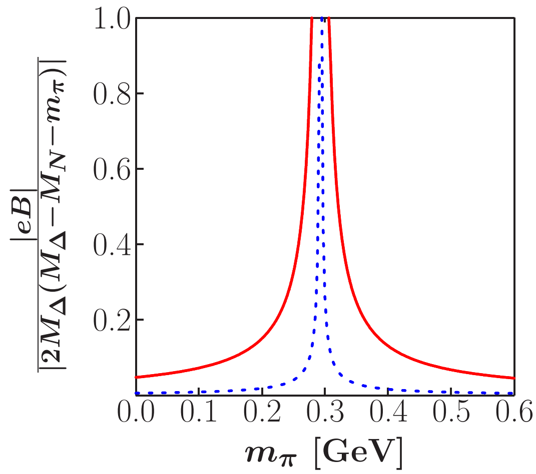

Figure 5: The condition Eq. (3) for the Δ-nucleon-pion system, $|\frac{eB}{2M_\Delta(M_\Delta - M_N - m_\pi)}| \ll 1$, plotted for the range of fields used in [10]: $|eB| = 0.00108/a^2 \ldots 0.00864/a^2$, with $1/a = 2$ GeV as function of the pion mass. Red (solid) curve corresponds to the stronger field and the blue (dashed) to the weaker field. The Delta-nucleon mass difference is taken to be constant: $M_\Delta - M_N = 0.293$ GeV.

One typical example would be the case of the $\Delta(1232)$ isobar, which magnetic moment has recently been computed using the background field method for various pion masses [10, 11]. Figure 5 shows how the condition

$$\left|\frac{eB}{2M_\Delta(M_\Delta - M_N - m_\pi)}\right| \ll 1 \qquad (17)$$

can be violated in this type of studies, but of course for very specific values of pion mass and the background magnetic field. We emphasize that the actual parameters in [10, 11], do not violate the above condition, mainly thanks to the large values of pion mass used in these works. However, current lattice calculations begin to approach the pion-mass range where this condition would be violated. It would be interesting to see how the non-analytic $B^{1/2}$ behavior emerges in these calculations. Of course one can expect this behavior to be shielded by the finite volume effects, the question is to which extent.

To conclude, the singularities found in calculations of the e.m. moments of particles, such as W-boson in the Standard Model (prior to the top-quark discovery) or some of the hadrons in chiral effective-field theory, reflect only the limitation of the calculational technique. When the mass of the particle is near a decay threshold (quasi-stable state), a small external e.m. field may induce the decay instead of interacting with the particle's e.m. moments. We have formulated an exact condition for this effect to occur. In this situation an extra care should be taken in defining and determining the moments, as has been described in this work. The present and future lattice QCD calculations of hadron e.m. moment using the background e.m. field technique are a very likely subject to this problem.

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